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2000 J. Phys. A: Math. Gen. 33 171

(http://iopscience.iop.org/0305-4470/33/1/311)

J. Phys. A: Math. Gen. **33** (2000) 171–189. Printed in the UK PII: S0305-4470(00)06559-8

Quantum Bäcklund transformation for the integrable DST model

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Received 3 August 1999

Abstract. For the integrable case of the discrete self-trapping (DST) model we construct a Bäcklund transformation. The dual Lax matrix and the corresponding dual Bäcklund transformation are also found and studied. The quantum analogue of the Bäcklund transformation (Q-operator) is constructed as the trace of a monodromy matrix with an infinite-dimensional auxiliary space. We present the Q-operator as an explicit integral operator as well as describing its action on the monomial basis. As a result we obtain a family of integral equations for multivariable polynomial eigenfunctions of the quantum integrable DST model. These eigenfunctions are special functions of the Heun class which is beyond the hypergeometric class. The integral equations found are new and they shall provide a basis for efficient analytical and numerical studies of such complicated functions.

1. Introduction

The discrete self-trapping (DST) equation was introduced by Eilbeck *et al* [1] to model the nonlinear dynamics of small molecules, such as ammonia, acetylene, benzene, as well as large molecules, such as acetanilide. In simple terms, it consists of a set of n nondissipative anharmonic oscillators coupled through dispersive interactions. Due to the nonlinearity this system can have complicated dynamical behaviour going from quasiperiodic motion to chaos [2, 3]. The DST equation is also found in connection with physical problems in different areas such as the stabilization of high-frequency vibrations in biomolecular dynamics [4], arrays of coupled nonlinear waveguides in nonlinear optics [5] and quasiparticle motion on a dimer [6]. In the case of two degrees of freedom $n = 2$ (DST dimer) the system is integrable having, besides the Hamiltonian (energy), another conserved quantity, the norm (number of particles in the quantum case). The integrability properties of the classical and quantum DST dimer were studied in detail by several methods such as the number state method [7], the algebraic Bethe ansatz [8] and the method of separation of variables [9]. For more than two degrees of freedom an integrable case of the DST system was found and studied in [10]. This integrable case is close to the Toda lattice and coincides for $n = 2$ with the usual DST dimer.

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0305-4470/00/010171+19$30.00

The quantum Hamiltonian, H, of the integrable DST model contains $(n+1)$ parameters $c_1, \ldots, c_n, b$ and is defined as a second-order differential operator (here, $\partial_i \equiv \partial/\partial x_i$)

$$H = \sum_{i=1}^{n} (\tfrac{1}{2} x_i^2 \partial_i^2 + (c_i + \tfrac{1}{2}) x_i \partial_i + b x_{i+1} \partial_i) \tag{1.1}$$

acting in the space $\mathbb{C}[\vec{x}]$ of polynomials of n variables $\{x_1, \ldots x_n\} \equiv \vec{x}$. In (1.1), and other similar formulae, we always assume the periodic boundary conditions $x_{n+1} \equiv x_1$.

The Hamiltonian H obviously commutes with the number-of-particles operator N

$$N = \sum_{i=1}^{n} x_i \partial_i. \tag{1.2}$$

As shown in section 4, H and N can be included in a commutative ring of differential operators generated by a basis of n operators, this fact allows one to claim the quantum integrability of the system.

The multiplication operators x_i and the respective differentiations ∂_i can be considered as generators of a Heisenberg algebra (creation/annihilation operators). There exists a well known scalar product on $\mathbb{C}[\vec{x}]$ (holomorphic representation) such that x_i and ∂_i become mutually adjoint $\partial_i^\dagger = x_i$. The corresponding Hamiltonian H is self-adjoint, however, only in the dimer case $n = 2$. In the general $n > 2$ case, no involution rendering H self-adjoint is known. The Hilbert space structure is, however, quite irrelevant for the kind of problems we are interested in and will be completely ignored throughout the paper.

The DST chain can be considered as a degenerate case of the Heisenberg magnetic chain, though not as degenerate as the Toda lattice. This makes the DST chain a good tool for studying various techniques applicable to integrable models since it requires more effort than the Toda lattice but is still simpler than the generic magnetic chain.

The main purpose of this paper is to construct an analogue of Baxter's Q-operator [11] for the integrable DST model. By definition, the Q-operator, Q_λ, shares the set of eigenvectors with the Hamiltonians H_i, and its eigenvalues are polynomials in λ satisfying a finite-difference equation known as the *Baxter* or *separation equation*. As was shown in [12] for the example of the periodic Toda lattice, in the classical limit the similarity transformation $\mathcal{O} \mapsto Q_\lambda \mathcal{O} Q_\lambda^{-1}$ turns into a classical Bäcklund transformation that is a one-parametric family of canonical transformations preserving the commuting Hamiltonians. Later, in [13], for the classical Bäcklund transfomations the property of *spectrality* was described which is the classical counterpart of the separation equation for the eigenvalues of Q_λ. In this paper we follow the approach of [13] first studying the classical case and paying special attention to the spectrality property of the corresponding Bäcklund transformation.

Our main result (see sections 4–7) is the following integral equation:

$$\int_\gamma \mathrm{d}\xi_1 \ldots \int_\gamma \mathrm{d}\xi_n \left[\prod_{i=1}^{n} \frac{\mathrm{i}}{2\pi} \Gamma(\lambda + 1 - c_i) \mathrm{e}^{-\xi_i} (-\xi_i)^{c_i - \lambda - 1} \psi \right] (\ldots, y_j \xi_j + b y_{j+1}, \ldots) \tag{1.3}$$

$$= q(\lambda) \psi(y_1, \ldots, y_n) \qquad q(\lambda) \in \mathbb{C}[\lambda] \tag{1.4}$$

for the polynomial eigenfunctions $\psi \in \mathbb{C}[\vec{x}]$ of the Hamiltonian (1.1)

$$H \psi(x_1, \ldots, x_n) = h \psi(x_1, \ldots, x_n). \tag{1.5}$$

The structure of this paper is as follows. In section 2 we consider the classical version of the integrable DST chain and describe its relation to the Toda lattice and the isotropic Heisenberg magnetic chain. Our construction of the Bäcklund transformation generalizes well known results for the Toda lattice. Following [13], we also study the dual Lax matrix and the corresponding dual Bäcklund transformation in section 3.

In section 4 we discuss the quantization of the integrable DST model and present a list of properties of Baxter's Q-operator. In section 5, following the approach of [14], we construct a Q-operator, Q_λ, for the quantum DST chain as the trace of a monodromy matrix with an infinite-dimensional auxiliary space. In the spirit of [12], we consider Q_λ as an integral operator in $\mathbb{C}[\vec{x}]$ and in section 6 find its kernel and contour of integration. In the same section we study analyticity properties of Q_λ in the parameter λ, prove that its matrix elements in the monomial basis are polynomials in λ and give explicit formulae for its action on polynomials. We consider in details the simplest $n = 1$ case where the Q-operator provides an integral representation for classical orthogonal polynomials (Charlier polynomials). In section 7 we prove that Q_λ satisfies a finite-difference equation in the parameter λ. Finally, in section 8, we discuss possible generalizations and applications of our results.

2. Classical case

In this section we consider the classical integrable DST chain [10]. The model is described in terms of n pairs of canonical variables (X_i, x_i), $i = 1, \dots, n$

$$\{X_i, X_j\} = \{x_i, x_j\} = 0 \qquad \{X_i, x_j\} = \delta_{ij} \tag{2.1}$$

(the periodicity convention $x_{i+n} \equiv x_i$, $X_{i+n} \equiv X_i$ is always assumed for the indices of x_i and X_i).

The canonical momenta X_i replace, in the classical case, the differential operators ∂_i. As mentioned before, in the quantum case we do not make any assumptions about the self-adjointness of the observables. Respectively, we allow the classical variables (X_i, x_i) to be complex.

To construct n commuting Hamiltonians we introduce the Lax matrix $L(u)$ (monodromy matrix) as a product of n local Lax matrices $\ell_i(u)$

$$L(u) = \ell_n(u) \dots \ell_2(u)\ell_1(u) \tag{2.2}$$

$$\ell_i(u; x_i, X_i) = \begin{pmatrix} u - c_i - x_i X_i & b x_i \\ -X_i & b \end{pmatrix} \tag{2.3}$$

where $b, c_i \in \mathbb{C}$ are parameters of the model, and u is the so-called spectral parameter of the Lax matrix.

Denoting by id_2 the unit 2×2 matrix and introducing notations for the tensor products $\overset{1}{\ell} \equiv \ell \otimes \mathrm{id}_2$, $\overset{2}{\ell} \equiv \mathrm{id}_2 \otimes \ell$ one establishes the r-matrix identity [15]

$$\{\overset{1}{\ell}_i(u_1), \overset{2}{\ell}_j(u_2)\} = [r_{12}(u_1 - u_2), \overset{1}{\ell}_i(u_1)\overset{2}{\ell}_j(u_2)]\delta_{ij} \qquad r_{12}(u) = -\frac{1}{u}\mathcal{P}_{12} \tag{2.4}$$

where $\mathcal{P}_{12}$ is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$. From (2.4) the corresponding identity for the monodromy matrix

$$\{\overset{1}{L}(u_1), \overset{2}{L}(u_2)\} = [r_{12}(u_1 - u_2), \overset{1}{L}(u_1)\overset{2}{L}(u_2)] \tag{2.5}$$

is derived in the standard way [15] which, in turn, ensures the commutativity of the spectral invariants $t(u)$ and $d(u)$ of the matrix $L(u)$ defined as coefficients of its characteristic polynomial

$$\det(v - L(u)) = v^2 - t(u)v + d(u). \tag{2.6}$$

Since $\det \ell_i(u) = b(u - c_i)$, the determinant $d(u) \equiv \det L(u) = \prod_{i=1}^n b(u - c_i)$ is scalar, and the only nontrivial spectral invariant is the trace $t(u)$:

$$t(u) \equiv \operatorname{tr} L(u) = L_{11}(u) + L_{22}(u) \tag{2.7}$$

which serves as a generating function of commuting independent Hamiltonians H_i:

$$t(u) = u^n + \sum_{i=1}^{n} (-1)^i H_i u^{n-i}. \tag{2.8}$$

As a corollary of (2.5) we have the commutativity of $t(u)$

$$\{t(u_1), t(u_2)\} = 0 \tag{2.9}$$

and, consequently, the commutativity $\{H_i, H_j\} = 0$ of the Hamiltonians H_i.

A direct calculation shows that

$$H_1 = N + \sum_{i=1}^{n} c_i \qquad H_2 = \tfrac{1}{2}H_1^2 - H - \tfrac{1}{2}\sum_{i=1}^{n} c_i^2 \tag{2.10}$$

where

$$N = \sum_{i=1}^{n} x_i X_i \qquad H = \sum_{i=1}^{n} (\tfrac{1}{2}x_i^2 X_i^2 + c_i x_i X_i + b x_{i+1} X_i) \tag{2.11}$$

ensuring that the polynomial ring of commuting Hamiltonians contains the number of particles N, and the Hamiltonian H.

Note that the r-matrix $r_{12}(u)$ in (2.4) is the same as for the isotropic Heisenberg magnetic chain and the Toda lattice [15], which puts these integrable models into the same class. Indeed, the Toda lattice is a degenerate case of the DST chain. To demonstrate this, it is sufficient to make a constant shift $u \mapsto u + b^{-1}$ of the spectral parameter in $\ell_i(u)$ given by (2.3) and take the limit

$$b \to 0 \qquad x_j = \mathrm{e}^{q_j}(b^{-1} + p_j) + \mathrm{O}(b) \qquad X_j = \mathrm{e}^{-q_j} \tag{2.12}$$

contracting the 'oscillator' algebra $(x_i, X_i, x_i X_i)$ into the Euclidean Lie algebra $(\mathrm{e}^{\pm q_i}, p_i)$. In the limit $\ell_i(u)$ turns into the elementary ℓ-matrix for the Toda lattice:

$$\ell_i(u) \to \begin{pmatrix} u - c_i - p_i & \mathrm{e}^{q_i} \\ -\mathrm{e}^{-q_i} & 0 \end{pmatrix} \tag{2.13}$$

(the c_i shifts become irrelevant since they can be absorbed into a simple canonical transformation $p_i \mapsto p_i - c_i$). On the other hand, the DST model, in turn, is a degenerate case of the Heisenberg XXX magnet corresponding to the contraction of the $su(2)$ Lie algebra into the oscillator algebra. The DST model occupies an intermediate position between the Heisenberg and Toda models.

In this paper we take the Hamiltonian point of view on the Bäcklund transformation, according to which the Bäcklund transformation B_λ is a one-parameter family of simplectic maps from the canonical variables $(\vec{X}, \vec{x})$ to the canonical variables $(\vec{Y}, \vec{y})$ possessing certain characteristic properties (see [13] for a detailed discussion). For Hamiltonian integrable systems allowing a description in terms of the r-matrix algebra (2.5) an algorithmic method has recently been found for constructing a Bäcklund transformation [16,17]. Since the method has been described in detail in the cited papers, here we present only the results.

As in the case of the periodic Toda lattice [12,13], it is convenient to describe the canonical transformation B_λ in terms of the generating function

$$F_\lambda(\vec{y} \mid \vec{x}) = n\lambda + \sum_{i=1}^{n} \left(\frac{x_i - b y_{i+1}}{y_i} + (\lambda - c_i) \ln \frac{b y_{i+1} - x_i}{(\lambda - c_i) b y_i} \right) \tag{2.14}$$

$$X_i = \frac{\partial F_\lambda}{\partial x_i} = \frac{1}{y_i} + \frac{\lambda - c_i}{x_i - b y_{i+1}} \tag{2.15a}$$

$$Y_i = -\frac{\partial F_\lambda}{\partial y_i} = b X_{i-1} + \frac{x_i - b y_{i+1}}{y_i} X_i. \tag{2.15b}$$

To prove that B_λ preserves the Hamiltonians H_i

$$H_i(\vec{X}, \vec{x}) = H_i(\vec{Y}, \vec{y}) \tag{2.16}$$

we proceed in the same manner as in [12, 13] for the periodic Toda lattice. Introducing the matrices

$$M_i(u) = \begin{pmatrix} 1 & -by_{i+1} \\ X_i & u - \lambda - by_{i+1}X_i \end{pmatrix} \tag{2.17}$$

one then directly verifies the equality

$$M_i(u)\ell_i(u; X_i, x_i) = \ell_i(u; Y_i, y_i)M_{i-1}(u) \tag{2.18}$$

from which it follows that B_λ preserves the spectrum of the Lax matrix $L(u)$

$$M_n(u, \lambda)L(u; \vec{X}, \vec{x}) = L(u; \vec{Y}, \vec{y})M_n(u, \lambda)$$

which, in turn, ensures the invariance of $t(u)$ and, therefore, of H_i (2.16).

To formulate the *spectrality property* [13] of the Bäcklund transformation we introduce the quantity μ canonically conjugated, in a sense, to λ:

$$\ln\mu = -\frac{\partial F_\lambda}{\partial\lambda} = \sum_{i=1}^{n} \ln\frac{(\lambda - c_i)by_i}{by_{i+1} - x_i} \qquad \mu = \prod_{i=1}^{n} \frac{(\lambda - c_i)by_i}{by_{i+1} - x_i}. \tag{2.19}$$

The spectrality of the Bäcklund transformation means that the (λ, μ) pair lies on the spectral curve of the Lax matrix

$$\det(\mu - L(\lambda)) = 0. \tag{2.20}$$

To prove it, we again follow [13]. We observe that for $u = \lambda$ the matrix $M_i(u)$ degenerates

$$M_i(\lambda) = \begin{pmatrix} 1 & -by_{i+1} \\ X_i & -by_{i+1}X_i \end{pmatrix} = \begin{pmatrix} 1 \\ X_i \end{pmatrix} \begin{pmatrix} 1 & -by_{i+1} \end{pmatrix} \tag{2.21}$$

and its null-vector ω_i can be found explicitly:

$$M_i(\lambda)\omega_i = 0 \qquad \omega_i = \begin{pmatrix} by_{i+1} \\ 1 \end{pmatrix}. \tag{2.22}$$

Then noting the identity

$$\ell_i(\lambda)\omega_{i-1} = \frac{(\lambda - c_i)by_i}{by_{i+1} - x_i}\omega_i \tag{2.23}$$

we conclude that

$$L(\lambda)\omega_n = \mu\omega_n \tag{2.24}$$

whence (2.20) follows immediately.

The commutativity $B_{\lambda_1} \circ B_{\lambda_2} = B_{\lambda_2} \circ B_{\lambda_1}$ is an immediate consequence of the invariance of Hamiltonians and their completeness, see [13].

Note that $M_i^{-1}(u)$ and $\ell_i(u)$ have, as functions of u, essentially the same structure, up to a shift of u and a scalar factor. The fact is by no means a coincidence: see [17] for a detailed explanation.

3. Dual Lax matrix

We conclude the study of the classical case by presenting the *dual* Lax matrix and the dual Bäcklund transformation for the DST model. In [10] two different Lax matrices were found for the integrable DST system, the 2×2 Lax matrix $L(u)$ and also the $n \times n$ Lax matrix. This bigger Lax matrix did not contain a spectral parameter. Here we present an $n \times n$ Lax matrix

$\mathcal{L}(v)$ containing a spectral parameter v which is dual to $L(u)$ in the sense that the corresponding spectral curves are equivalent up to interchanging the spectral parameters u and v

$$(b^n - v)\det(u - \mathcal{L}(v)) = \det(v - L(u)). \tag{3.1}$$

To produce the dual Lax matrix $\mathcal{L}(v)$ we take an eigenvector $\theta_1(u)$ of $L(u)$ corresponding to the eigenvalue v (for brevity, we will not mark the dependence on u in θ)

$$L(u)\theta_1 = v\theta_1 \tag{3.2}$$

and define by induction θ_i as

$$\theta_{i+1} = \ell_i(u)\theta_i \qquad i = 1, \ldots, n. \tag{3.3}$$

From (3.2) it follows that $\theta_{n+1} = v\theta_1$. The function $\theta_i(u)$, when properly normalized, is called the *Baker–Akhiezer function*. Denoting the components of the vector θ_i as φ_i and ψ_i, we present (3.3) explicitly as

$$\begin{pmatrix} \varphi_{i+1} \\ \psi_{i+1} \end{pmatrix} = \begin{pmatrix} u - c_i - x_i X_i & b x_i \\ -X_i & b \end{pmatrix} \begin{pmatrix} \varphi_i \\ \psi_i \end{pmatrix}. \tag{3.4}$$

Then, splitting the components and taking into account the quasiperiodicity condition $\theta_{n+1} = v\theta_1$, we arrive at the following linear equations for φ_i and ψ_i:

$$u\varphi_i = \varphi_{i+1} + (c_i + x_i X_i)\varphi_i - b x_i \psi_i \qquad i = 1, \ldots, n-1 \tag{3.5a}$$

$$u\varphi_n = v\varphi_1 + (c_n + x_n X_n)\varphi_n - b x_n \psi_n \tag{3.5b}$$

$$\psi_{i+1} = -X_i\varphi_i + b\psi_i \qquad i = 1, \ldots, n-1 \tag{3.6a}$$

$$v\psi_1 = -X_n\varphi_n + b\psi_n. \tag{3.6b}$$

Eliminating ψ_i we can write down the linear problem for the vector Φ with the components φ_i in the matrix form:

$$\mathcal{L}(v)\Phi = u\Phi \qquad \Phi = \begin{pmatrix} \varphi_1 \\ \ldots \\ \varphi_n \end{pmatrix} \tag{3.7}$$

where the matrix $\mathcal{L}(v)$ defined as

$$\begin{aligned}\mathcal{L}(v) = (v - b^n)^{-1} \sum_{j,k=1}^{n} b^{n+j-k} x_j X_k E_{jk} + v E_{n1} + \sum_{j \geqslant k} b^{j-k} x_j X_k E_{jk} \\ + \sum_{j=1}^{n} c_j E_{jj} + \sum_{j=1}^{n-1} E_{j,j+1}\end{aligned} \tag{3.8}$$

is the dual Lax matrix we were looking for. Here E_{jk} is the $n \times n$ matrix with the only non-zero entry $(E_{jk})_{jk} = 1$. The proof of identity (3.1) is an exercise which we leave to the reader. For the case $b = 1$ and $v = -1$ the dual Lax matrix for the DST model was first found in [10]. For examples of Lax matrices duality in other integrable models see [18].

The Bäcklund transformation $\mathcal{B}_\mu$ corresponding to the dual Lax operator $\mathcal{L}(v)$ is given by the same equations: (2.15*a*), (2.15*b*) and (2.19). The important difference, however, is that now μ is a free numerical parameter of the Bäcklund transformation, whereas λ becomes a dynamical variable determined from equation (2.19). Equality (2.19) is now reinterpreted as the equation defining the variable λ. The generating function of $\mathcal{B}_\mu$ is the Legendre transform of $F_\lambda(\vec{y} \mid \vec{x})$ with respect to λ.

The properties of the dual Bäcklund transformation $\mathcal{B}_\mu$ are proved in the same manner as those of B_λ (see also [13], for the Toda lattice case). For the proof we need a matrix $\mathcal{M}(v)$ playing for $\mathcal{L}(v)$ the same role that $M_n(u)$ played for $L(u)$.

Let $\tilde{\theta}_i$ be defined as $\tilde{\theta}_i = M_{i-1}\theta_i$. From (2.18) it follows that $\tilde{\theta}_i$ is a Baker–Akhiezer function for $\ell_i(u; Y_i, y_i)$. The first component of the equality $\tilde{\theta}_i = M_{i-1}\theta_i$ reads $\tilde{\varphi}_i = \varphi_i - by_i\psi_i$. Substituting ψ_i from the solution of the system (3.6*a*), (3.6*b*) we obtain the correspondence $\tilde{\Phi} = \mathcal{M}(v)\Phi$ with the matrix $\mathcal{M}(v)$ defined as

$$\mathcal{M}(v) = (v - b^n)^{-1}\sum_{j,k=1}^{n} b^{n+j-k} y_j X_k E_{jk} + \sum_{j>k} b^{j-k} y_j X_k E_{jk} + \sum_{j=1}^{n} E_{jj}. \tag{3.9}$$

The invariance of the spectrum of $\mathcal{L}(v)$ follows from the identity

$$\mathcal{M}(v)\mathcal{L}(v; \vec{X}, \vec{x}) = \mathcal{L}(v; \vec{Y}, \vec{y})\mathcal{M}(v). \tag{3.10}$$

The spectrality is expressed as the identity

$$\det(\lambda - \mathcal{L}(\mu)) = 0. \tag{3.11}$$

To prove (3.11) it is sufficient to note that the matrix $\mathcal{M}(v)$ degenerates as $v = \mu$

$$\det \mathcal{M}(\mu) = 0 \tag{3.12}$$

and the corresponding null-vector Ω defined by the recurrence relation

$$\frac{\Omega_{i+1}}{\Omega_i} = \frac{b(c_i - \lambda)y_{i+1}}{x_i - by_{i+1}} \qquad i = 1, \ldots, n-1 \tag{3.13}$$

is, by virtue of (3.10), also an eigenvector of $\mathcal{L}(\mu)$ corresponding to the eigenvalue λ

$$\mathcal{L}(\mu)\Omega = \lambda\Omega. \tag{3.14}$$

Since the Toda lattice is a degenerate case of the DST model, the $n \times n$ Lax matrix for the Toda lattice can be obtained, as one could expect, from our $\mathcal{L}(v)$ matrix in the limit $b \to 0$, as in (2.12). The result is a variant of the standard $n \times n$ Lax matrix for the periodic Toda lattice [19]:

$$\mathcal{L}(v) = b^{-1} + \mathcal{L}^{TL}(v) + \mathrm{O}(b) \tag{3.15}$$

$$\mathcal{L}^{TL}(v) = v^{-1}\mathrm{e}^{q_n - q_1} E_{1n} + vE_{n1} + \sum_{j=1}^{n}(p_j + c_j)E_{jj} + \sum_{j=1}^{n-1}\mathrm{e}^{q_j - q_{j+1}} E_{j+1,j} + \sum_{j=1}^{n-1} E_{j,j+1}. \tag{3.16}$$

Similarly, from $\mathcal{M}(v)$ one obtains the corresponding matrix for the Toda lattice, see [13].

The Poisson brackets for both dual Lax matrices $\mathcal{L}(v)$ can be expressed in the generalized r-matrix form [20]

$$\{\overset{1}{\mathcal{L}}(v_1), \overset{2}{\mathcal{L}}(v_2)\} = [r_{12}(v_1, v_2), \overset{1}{\mathcal{L}}(v_1)] - [r_{21}(v_1, v_2), \overset{2}{\mathcal{L}}(v_2)] \tag{3.17}$$

the 'non-unitary' r-matrix having the form

$$r_{12}(v_1, v_2) = \frac{1}{v_1 - v_2}\left(v_2\sum_{k\geqslant j} + v_1\sum_{k<j}\right)E_{jk} \otimes E_{kj} \tag{3.18}$$

and $r_{21}(v_1, v_2) = \mathcal{P}r(v_2, v_1)\mathcal{P}$, where $\mathcal{P} = \sum_{j,k=1}^{n} E_{jk} \otimes E_{kj}$ is the permutation matrix in $\mathbb{C}^n \otimes \mathbb{C}^n$.

The non-unitary r-matrix (3.18) in the case of Toda's Lax matrix can be unitarized by a gauge transformation:

$$\mathfrak{L}(v) = V\mathcal{L}^{TL}(v)V^{-1} \qquad V = \sum_{j=1}^{n}\mathrm{e}^{q_j/2} E_{jj} \tag{3.19}$$

obtaining, for the new Lax matrix $\mathfrak{L}(v)$, the standard unitary A_{n-1}-type r-matrix

$$\mathfrak{r}_{12}(v_1, v_2) = \frac{v_1 + v_2}{v_1 - v_2} \sum_{j=1}^{n} E_{jj} \otimes E_{jj} + \frac{1}{v_1 - v_2}\left(v_2 \sum_{k>j} + v_1 \sum_{k<j}\right) E_{jk} \otimes E_{kj} \tag{3.20}$$

$$\mathfrak{r}_{12}(v_1, v_2) = -\mathfrak{r}_{21}(v_1, v_2) \tag{3.21}$$

$$\{\overset{1}{\mathfrak{L}}(v_1), \overset{2}{\mathfrak{L}}(v_2)\} = [\mathfrak{r}_{12}(v_1, v_2), \overset{1}{\mathfrak{L}}(v_1) + \overset{2}{\mathfrak{L}}(v_2)] \tag{3.22}$$

see, for instance, the second paper in [19].

4. Quantization

In the quantum case the canonical momenta X_i are replaced with the differentiations $\partial_i \equiv \partial/\partial x_i$ (having no intent to discuss the conjugation properties of the observables, we discard the factor $\mathrm{i}\hbar$ to simplify the notation). To preserve the commutativity of the Hamiltonians H_i upon quantization one needs to choose the operator ordering in a special way.

The necessary algebraic framework is given by the quantum inverse scattering or the R-matrix [11, 21] method. Defining the local quantum Lax matrix as

$$\ell_i(u) = \begin{pmatrix} u - c_i - x_i\partial_i & bx_i \\ -\partial_i & b \end{pmatrix} \tag{4.1}$$

one verifies the commutation relation

$$R_{12}(u_1 - u_2)\overset{1}{\ell}(u_1)\overset{2}{\ell}(u_2) = \overset{2}{\ell}(u_2)\overset{1}{\ell}(u_1)R_{12}(u_1 - u_2) \tag{4.2}$$

where

$$R_{12}(u) = u + \mathcal{P}_{12} \tag{4.3}$$

is the standard $SL(2)$-invariant solution to the quantum Yang–Baxter equation. The quantum Lax operator, or monodromy matrix, $L(u)$ and its trace $t(u)$ are defined then by the same formulae, (2.2) and (2.7), as in the classical case. From (4.2) one then derives in a standard way the similar commutation relation

$$R_{12}(u_1 - u_2)\overset{1}{L}(u_1)\overset{2}{L}(u_2) = \overset{2}{L}(u_2)\overset{1}{L}(u_1)R_{12}(u_1 - u_2) \tag{4.4}$$

for $L(u)$, from which the commutativity of $t(u)$

$$[t(u_1), t(u_2)] = 0 \tag{4.5}$$

follows immediately. The commutative quantum Hamiltonians H_i are then defined, as in the classical case (2.8), as coefficients of the polynomial $t(u)$. It is easy to see that H_i is a differential operator of order i leaving invariant the space $\mathbb{C}[\vec{x}]$ of polynomials of $x_1, \ldots x_n$. In particular, H_1 and H_2 are given by the formulae (2.10) with N and H given by (1.2) and (1.1), respectively.

The main problem in the quantum case is the spectral problem for commuting differential operators, quantum Hamiltonians $\{H_i\}_{i=1}^n$:

$$H_i\psi(x_1, \ldots, x_n) = h_i\psi(x_1, \ldots, x_n) \qquad \psi(x_1, \ldots, x_n) \in \mathbb{C}[\vec{x}]. \tag{4.6}$$

One can describe the spectrum and eigenvectors of H_i, or, equivalently, $t(u)$ using the well-developed machinery of the *algebraic Bethe ansatz* [21]. Defining the vacuum state $|0\rangle$ as the unit function $|0\rangle(x) \equiv 1$ in $\mathbb{C}[\vec{x}]$ we note that

$$L_{21}|0\rangle = 0 \qquad L_{11}(u)|0\rangle = \alpha_{11}(u)|0\rangle \qquad L_{22}(u)|0\rangle = \alpha_{22}(u)|0\rangle \tag{4.7}$$

where

$$\alpha_{11}(u) = \prod_{i=1}^{n}(u - c_i) \qquad \alpha_{22}(u) = b^n. \tag{4.8}$$

Defining the *Bethe vector* $\psi_{\vec{v}}(x_1, \dots, x_n) \in \mathbb{C}[\vec{x}]$ parametrized by m complex numbers v_j as

$$\psi_{\vec{v}}(x_1, \dots, x_n) \equiv |v_1, \dots, v_m\rangle = L_{12}(v_1) \dots L_{12}(v_m)|0\rangle \tag{4.9}$$

one can prove [21], using the commutation relations (4.4), that $|v_1, \dots, v_m\rangle$ is an eigenvector of $t(u)$, for any $u \in \mathbb{C}$, if and only if the parameters v_j satisfy the system of algebraic *Bethe equations*

$$\prod_{j=1}^{m} \frac{v_k - v_j + 1}{v_k - v_j - 1} = -\frac{\alpha_{11}(v_k)}{\alpha_{22}(v_k)} \qquad k = 1, \dots, m \tag{4.10}$$

and the corresponding eigenvalue $\tau(u)$ of $t(u)$

$$t(u)|v_1, \dots, v_m\rangle = \tau(u)|v_1, \dots, v_m\rangle \tag{4.11}$$

is given by the formula

$$\tau(u) = \alpha_{11}(u) \prod_{j=1}^{m} \frac{u - v_j - 1}{u - v_j} + \alpha_{22}(u) \prod_{j=1}^{m} \frac{u - v_j + 1}{u - v_j}. \tag{4.12}$$

It is usually assumed that Bethe eigenvectors are complete, at least for generic values of parameters. The proof of the conjecture is, however, a difficult task, and is available only for a few models, see [21] for a discussion.

In his seminal study [11] of the integrable XYZ and XXZ spin chains R J Baxter has pointed out that the equations similar to our equations (4.10) and (4.12) can be reformulated equivalently as a finite-difference equation in a certain class of holomorphic functions. Adapting his reasoning to our case we introduce the polynomial $\phi(\lambda; \vec{v})$ in λ whose zeros are the Bethe parameters v_j:

$$\phi(\lambda; \vec{v}) = \prod_{j=1}^{m} (\lambda - v_j) \qquad \lambda \in \mathbb{C}. \tag{4.13}$$

It is then easy to see that the following finite-difference equation of second order for $\phi(\lambda; \vec{v})$:

$$\phi(\lambda; \vec{v})\tau(\lambda) = \alpha_{11}(\lambda)\phi(\lambda - 1; \vec{v}) + \alpha_{22}(\lambda)\phi(\lambda + 1; \vec{v}) \tag{4.14}$$

is equivalent to the system of equations (4.10) for $\{v_j\}_{j=1}^m$, and to equation (4.12) for $\tau(\lambda)$. To show this, it is sufficient to divide both sides of (4.14) by $\phi(\lambda)$ and take residues at $\lambda = v_j$. The equation (4.14) is called the *Baxter* or *separation* equation. The reason for the latter name is that an identical equation arises when solving the model via the separation of variables method (see [13] for more on relation between Q-operator and quantum separation of variables).

Now we are able to describe the problem we are going to study in the remaining sections of this paper. We are looking for a one-parameter family of operators Q_λ acting in $\mathbb{C}[\vec{x}]$ such that Q_λ shares with $t(u)$ the same set of Bethe eigenvectors, and the eigenvalues $q(\lambda)$ of Q_λ

$$Q_\lambda|v_1, \dots, v_m\rangle = q(\lambda)|v_1, \dots, v_m\rangle \tag{4.15}$$

are polynomials in λ satisfying Baxter's equation (4.14). Up to a normalization coefficient $\kappa_{\vec{v}}$, depending on the eigenvector, the polynomials $q(\lambda)$ are proportional to the polynomials $\phi(\lambda; \vec{v})$ defined by (4.13):

$$q(\lambda) = \kappa_{\vec{v}}\phi(\lambda; v_1, \dots, v_m) = \kappa_{\vec{v}}\lambda^m + \mathrm{O}(\lambda^{m-1}) \qquad \lambda \to \infty. \tag{4.16}$$

Instead of dealing with eigenvectors and eigenvalues it is more convenient to characterize Q_λ by the following operator identities which are equivalent to the above characterization, assuming the completeness of Bethe eigenvectors. We demand that Q_λ commute with $t(u)$

$$[t(u), Q_\lambda] = 0 \tag{4.17a}$$

and self-commute

$$[Q_{\lambda_1}, Q_{\lambda_2}] = 0 \tag{4.17b}$$

as well as satisfy the finite-difference equation

$$Q_\lambda t(\lambda) = Q_{\lambda-1} \prod_{i=1}^{n} (\lambda - c_i) + b^n Q_{\lambda+1}. \tag{4.17c}$$

In addition, the eigenvalues of Q_λ should be polynomial in λ

$$q(\lambda) \in \mathbb{C}[\lambda]. \tag{4.17d}$$

The above conditions by no means define Q_λ uniquely. Apparently, one can construct infinitely many Q-operators just by fixing arbitrary normalization coefficients $\kappa_{\vec{v}}$ for each eigenvector $|\vec{v}\rangle$ in (4.16). The difficult problem is to find an explicit expression for a Q-operator. Baxter succeeded in solving the problem in case of XYZ and XXZ spin chains, having given an expression for Q_λ as a trace of a monodromy matrix [11]. However, his formulae do not survive when passing to the limiting case of the XXX spin chain, governed by the $SL(2)$ invariant R-matrix (4.3).

In the case of the quantum periodic Toda lattice, which is another model governed by the R-matrix (4.3), a solution was found by Pasquier and Gaudin [12]. Instead of trying to construct Q_λ as trace of a monodromy matrix, they considered Q_λ as an integral operator

$$Q_\lambda : \psi(\vec{x}) \mapsto \int \mathrm{d}x_1 \ldots \int \mathrm{d}x_n \, \mathcal{Q}_\lambda(\vec{y} \mid \vec{x})\psi(\vec{x}) \tag{4.18}$$

having given an explicit expression for its kernel $\mathcal{Q}_\lambda(\vec{y} \mid \vec{x})$. They also discovered an important relation between the kernel $\mathcal{Q}_\lambda(\vec{y} \mid \vec{x})$ and the generating function $F_\lambda(\vec{y} \mid \vec{x})$ of the classical Bäcklund transformation expressed by the semiclassical formula

$$\mathcal{Q}_\lambda(\vec{y} \mid \vec{x}) \sim \exp\left(-\frac{\mathrm{i}}{\hbar} F_\lambda(\vec{y} \mid \vec{x})\right) \qquad \hbar \to 0. \tag{4.19}$$

The classical Bäcklund transformation B_λ is thus the classical limit of the similarity transformation $\mathcal{O} \mapsto Q_\lambda \mathcal{O} Q_\lambda^{-1}$.

Recently, it was found [14] how the original Baxter's construction [11] can be generalized to produce Q-operators for the models governed by the A_1-type R-matrices, such as the XXZ spin chain and sine–Gordon model. According to [14], Q_λ is constructed as the trace of a monodromy matrix built from the local Lax operators corresponding, in the auxiliary space, to the special infinite-dimensional representations of the quantum group $\mathcal{U}_q[\widehat{sl_2}]$ (*q-oscillator* representations).

In the subsequent sections we construct a Q-operator for the quantum DST model and prove its characteristic properties. Our approach combines those of [12, 14]. Similarly to [14], we construct our Q-operator as the trace of a monodromy matrix with an infinite-dimensional auxiliary space. In the spirit of [12], we find Q_λ as an integral operator acting in $\mathbb{C}[\vec{x}]$ and present several equivalent expressions for it.

The Q-operator being found as an integral operator will give integral equations for the eigenfunctions $\psi_{\vec{v}}$. The advantage of this transformation of the differential spectral problem into integral spectral problem is that it gives an alternative to the Bethe representation of multivariable special functions. The general approach of constructing a Q-operator for a given integrable system will be of even greater importance in situations when the Bethe ansatz does not work.

5. Construction of the Q-operator

The structure of Q_λ is similar to that of $t(u)$ given by (2.2) and (2.7). We construct Q_λ as the trace of a monodromy matrix built of the elementary blocks $\mathbb{R}^{(i)}_{\lambda-c_i}$. Suppose that $\mathbb{R}_\lambda$ is a linear operator from $\mathbb{C}[s, x]$ to $\mathbb{C}[t, y]$. The spaces $\mathbb{C}[x]$ and $\mathbb{C}[y]$ are referred to as *quantum* spaces and $\mathbb{C}[s]$ and $\mathbb{C}[t]$, respectively, as *auxiliary* ones (see [21]). To construct Q_λ we introduce n copies $\mathbb{R}^{(i)}_{\lambda-c_i}$ of $\mathbb{R}_\lambda$ assuming that $\mathbb{R}^{(i)}_{\lambda-c_i} : \mathbb{C}[s_i, x_i] \mapsto \mathbb{C}[s_{i+1}, y_i]$ (remember the periodicity convention, $n+1 \equiv 1$) and extending $\mathbb{R}^{(i)}_{\lambda-c_i}$ on $\mathbb{C}[x_j]$ ($j \neq i$) as the unit operator. The monodromy matrix $\mathbb{R}^{(n)}_{\lambda-c_n} \dots \mathbb{R}^{(1)}_{\lambda-c_1}$ then acts from $\mathbb{C}[s_1, \vec{x}]$ into $\mathbb{C}[s_1, \vec{y}]$, and Q_λ is obtained by taking the trace in the auxiliary space $\mathbb{C}[s_1]$:

$$Q_\lambda = \mathrm{tr}_{s_1} \mathbb{R}^{(n)}_{\lambda-c_n} \dots \mathbb{R}^{(1)}_{\lambda-c_1}. \tag{5.1}$$

Supposing $\mathbb{R}_\lambda$ to be an integral operator

$$\mathbb{R}_\lambda : \psi(s, x) \mapsto \int \mathrm{d}x \int \mathrm{d}s\, \mathcal{R}_\lambda(t, y \mid s, x)\psi(s, x) \tag{5.2}$$

for the kernel $\mathcal{Q}_\lambda(\vec{y} \mid \vec{x})$ of Q_λ we have

$$\mathcal{Q}_\lambda(\vec{y} \mid \vec{x}) = \int \mathrm{d}s_n \dots \int \mathrm{d}s_1 \prod_{i=1}^{n} \mathcal{R}_{\lambda-c_i}(s_{i+1}, y_i \mid s_i, x_i). \tag{5.3}$$

To ensure the commutativity $[t(u), Q_\lambda] = 0$ it is sufficient to demand that $\mathbb{R}_\lambda$ intertwines

$$\mathcal{M}(u-\lambda; t, \partial_t)\ell(u; y, \partial_y)\mathbb{R}_\lambda = \mathbb{R}_\lambda \ell(u; x, \partial_x)\mathcal{M}(u-\lambda; s, \partial_s) \tag{5.4}$$

the local Lax operator $\ell(u)$ and some other representation $\mathcal{M}(u-\lambda)$ of the same algebra (4.2)

$$R(u_1-u_2)\overset{1}{\mathcal{M}}(u_1)\overset{2}{\mathcal{M}}(u_2) = \overset{2}{\mathcal{M}}(u_2)\overset{1}{\mathcal{M}}(u_1)R(u_1-u_2) \tag{5.5}$$

with the same R-matrix (4.3). The proof of (4.17*a*) then follows by a standard argument [15,21].

Similarly, to prove $[Q_{\lambda_1}, Q_{\lambda_2}] = 0$ (4.17*b*) it is sufficient to establish the Yang–Baxter identity

$$\begin{aligned}\int \mathrm{d}t_1 \int \mathrm{d}t_2 \int \mathrm{d}y\, \tilde{\mathcal{R}}_{\lambda_1-\lambda_2}(w_1, w_2 \mid t_1, t_2)\mathcal{R}_{\lambda_1}(t_1, z \mid s_1, y)\mathcal{R}_{\lambda_2}(t_2, y \mid s_2, x)\\ = \int \mathrm{d}t_1 \int \mathrm{d}t_2 \int \mathrm{d}y\, \mathcal{R}_{\lambda_2}(w_2, z \mid t_2, y)\mathcal{R}_{\lambda_1}(w_1, y \mid t_1, x)\tilde{\mathcal{R}}_{\lambda_1-\lambda_2}(t_1, t_2 \mid s_1, s_2)\end{aligned} \tag{5.6}$$

with some kernel $\tilde{\mathcal{R}}_\lambda$.

The representation $\mathcal{M}(u-\lambda)$ of the algebra (5.5) should be chosen in such a way that the resulting Q_λ, as a function of λ, satisfy Baxter's finite-difference equation (4.17*c*) and have polynomial eigenvalues (4.17*d*). As we shall show, for this purpose one can take

$$\mathcal{M}(u; s, \partial_s) = \begin{pmatrix} u - s\partial_s & s \\ -\partial_s & 1 \end{pmatrix} \tag{5.7}$$

coinciding essentially with $\ell(u)$ with $b = 1$ and $c_i = 0$. For the Yangian $\mathcal{Y}[sl_2]$ representation (5.7) plays the same role as the q-oscillator representation plays for the quantum group $\mathcal{U}_q[\widehat{sl_2}]$ in [14]. Having fixed $\mathcal{M}(u)$ by (5.7) we get, from (5.4), a set of differential equations for the kernel $\mathcal{R}_\lambda(t, y \mid s, x)$ of $\mathbb{R}_\lambda$

$$\begin{aligned}\begin{pmatrix} u-\lambda-t\partial_t & t \\ -\partial_t & 1 \end{pmatrix}\begin{pmatrix} u-y\partial_y & by \\ -\partial_y & b \end{pmatrix}\mathcal{R}_\lambda(t, y \mid s, x)\\ = \begin{pmatrix} u+1+x\partial_x & bx \\ \partial_x & b \end{pmatrix}\begin{pmatrix} u-\lambda+1+s\partial_s & s \\ \partial_s & 1 \end{pmatrix}\mathcal{R}_\lambda(t, y \mid s, x)\end{aligned} \tag{5.8}$$

(in the right-hand side we have used integration by parts and the identities $\partial_x^* = -\partial_x$, $(x\partial_x)^* = -\partial_x x = -1 - x\partial_x$). Equations (5.8) determine $\mathcal{R}_\lambda$ up to a scalar factor ρ_λ:

$$\mathcal{R}_\lambda(t, y \mid s, x) = \rho_\lambda \delta(s-by)y^{-1} \exp\left(\frac{t-x}{y}\right)\left(\frac{t-x}{y}\right)^{-\lambda-1}. \tag{5.9}$$

It remains to choose the factor ρ_λ in (5.9) and the integration contour in (5.2) in such a way that $\mathbb{R}_\lambda : \mathbb{C}[s, x] \mapsto \mathbb{C}[t, y, \lambda]$.

We describe first the final formula for $\mathbb{R}_\lambda$ and equivalent expressions and then prove the polynomiality property. As the basic definition of $\mathbb{R}_\lambda$ we choose the following formula:

$$\mathbb{R}_\lambda : \psi(s, x) \mapsto \frac{\mathrm{i}}{2\pi}\Gamma(\lambda + 1)\int_\gamma \mathrm{d}\xi\, \mathrm{e}^{-\xi}(-\xi)^{-\lambda-1}\psi(by, y\xi + t). \tag{5.10}$$

The infinite integration contour γ is shown in figure 1. The branch of the many-valued function $(-\xi)^{-\lambda-1}$ in (5.10) is fixed by making a cut along $(0, \infty)$ and assuming that $-\pi \leqslant \arg(-\xi) \leqslant \pi$.

From (5.10) it is apparent that $\mathbb{R}_\lambda$, as a function of λ, is analytic in $\mathbb{C}$ except at the poles $\lambda = -1, -2, \ldots$ of the factor $\Gamma(\lambda + 1)$. As shown below, in fact $\mathbb{R}_\lambda$ continues analytically on the whole complex plane.

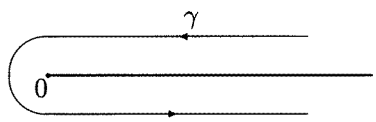

Figure 1. Integration contour γ.

Indeed, for $\mathrm{Re}\,\lambda < 0$ one can pull the contour γ over the cut $(0, \infty)$ and replace $\int_\gamma \mathrm{d}\xi\ f(\xi)$ with $\int_0^\infty \mathrm{d}\xi[f(\xi - \mathrm{i}0) - f(\xi + \mathrm{i}0)]$ which results in the formula

$$\mathbb{R}_\lambda : \psi(s, x) \mapsto \frac{1}{\Gamma(-\lambda)}\int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-\xi}\xi^{-\lambda-1}\psi(by, y\xi + t) \qquad \mathrm{Re}\,\lambda < 0 \tag{5.11}$$

which is analytic in $\lambda = -1, -2, \ldots$. The branch of $\xi^{-\lambda-1}$ in (5.11) is fixed by the condition $\arg \xi = 0$.

To put $\mathbb{R}_\lambda$ in the form (5.2) convenient for checking the intertwining relation (5.8) one has to make the change of variables $x = y\xi + t$ in (5.10). The result is given by formula (5.2) with the kernel $\mathcal{R}_\lambda$ given by expression (5.9) with the scalar factor $\rho_\lambda = \frac{\mathrm{i}}{2\pi}\Gamma(\lambda + 1)$ and integration in x taken over the contour $\gamma' = y\gamma + t$. As for the integration contour in s, it needs only to pass through the point $s = by$ because of the factor $\delta(s - by)$ in $\mathcal{R}_\lambda$.

In the same way, from (5.11) one again obtains formula (5.2) with kernel (5.9) with the different scalar factor $\rho_\lambda = 1/\Gamma(-\lambda)$ and integration in x taken over the ray starting from $x = t$ and going in the direction of $y/|y|$.

Now we have the full description of the operator $\mathbb{R}_\lambda$ and can start to study its properties. By construction, $\mathbb{R}_\lambda$ satisfies relation (5.4) from which the commutativity $[t(u), Q_\lambda] = 0$ (4.17*a*) follows. By direct calculation one can also establish the Yang–Baxter identity (5.6) with the kernel $\tilde{\mathcal{R}}_\lambda \equiv \mathcal{R}_\lambda|_{b=1}$ thus proving the commutativity $[Q_{\lambda_1}, Q_{\lambda_2}] = 0$ (4.17*b*). The proof of the remaining properties of Q_λ from the list presented in section 4 is given in sections 6 and 7.

We conclude this section by giving an alternative description of $\mathbb{R}_\lambda$ in terms of the polynomial bases which complements the above ones in terms of integral operators.

To calculate explicitly the action of $\mathbb{R}_\lambda$ on the monomial basis $s^k x^j$ in $\mathbb{C}[s, x]$ one puts $\psi(s, x) = s^k x^j$ in (5.10), then expands the binomial $(y\xi + t)^j$ and applies, termwise, Hankel's integral formula [22]

$$\int_\gamma \mathrm{d}\xi\, \mathrm{e}^{-\xi}(-\xi)^{\nu-1} = -\frac{2\pi\mathrm{i}}{\Gamma(1 - \nu)}. \tag{5.12}$$

Using the Pochhammer symbol $(c)_m \equiv \Gamma(c+m)/\Gamma(c) = c(c+1)\dots(c+m-1)$ one can write down the result as

$$\mathbb{R}_\lambda : s^k x^j \mapsto \sum_{m=0}^{j} \binom{j}{m} (-\lambda)_m t^{j-m} y^{m+k} b^k = t^j b^k C_j(\lambda; t/y) \tag{5.13}$$

where $C_m(\lambda; b)$ are the so-called *Charlier polynomials* [22, 23]

$$C_m(\lambda; b) = {}_2F_0 \left[\begin{matrix} -m, -\lambda \\ - \end{matrix} ; -b^{-1} \right].$$

Formula (5.13) proves the polynomiality $\mathbb{R}_\lambda : \mathbb{C}[s, x] \mapsto \mathbb{C}[t, y, \lambda]$. Note that the normalization of $\mathbb{R}_\lambda$ is chosen in such a way that $\mathbb{R}_\lambda : 1 \mapsto 1$.

The action of $\mathbb{R}_\lambda$ on polynomials can be described in an even more compact way. Substituting $\psi(s, x) = s^k (x - t)^j$ into (5.10) and using again Hankel's formula (5.12) one obtains the most economic description of $\mathbb{R}_\lambda$

$$\mathbb{R}_\lambda : s^k (x - t)^j \mapsto y^{j+k} (-\lambda)_j b^k. \tag{5.14}$$

At the end of the next section we will discuss (5.14) and similar formulae in more detail.

6. Analytical properties of the Q-operator

To produce a description of Q_λ as an integral operator (4.18) we substitute expression (5.9) for the kernel $\mathcal{R}_\lambda$ found in the previous section into formula (5.3). The integration in s_i is easily performed due to the delta-function factors in $\mathcal{R}_\lambda$ and, corresponding to the two choices of the factor ρ_λ in (5.9) and the integration contour in (5.2), we obtain two equivalent descriptions of Q_λ.

The first formula for Q_λ is given by (4.18) with the kernel $\mathcal{Q}_\lambda(\vec{y} \mid \vec{x})$

$$\mathcal{Q}_\lambda(\vec{y} \mid \vec{x}) = \prod_{i=1}^{n} w_i(\lambda; y_{i+1}, y_i, x_i) \tag{6.1}$$

where

$$w_i(\lambda; y_{i+1}, y_i, x_i) = \frac{\mathrm{i}}{2\pi} \Gamma(\lambda + 1 - c_i) y_i^{-1} \left(\frac{by_{i+1} - x_i}{y_i} \right)^{c_i - \lambda - 1} \exp\left(\frac{by_{i+1} - x_i}{y_i} \right) \tag{6.2}$$

and integration in x_i is taken over the contour $\gamma_i = y_i \gamma + by_{i+1}$, whereas the contour γ is defined in the previous section.

The alternative formula is given again by (4.18) with the kernel $\tilde{\mathcal{Q}}_\lambda(\vec{y} \mid \vec{x})$

$$\tilde{\mathcal{Q}}_\lambda(\vec{y} \mid \vec{x}) = \prod_{i=1}^{n} \tilde{w}_i(\lambda; y_{i+1}, y_i, x_i) \tag{6.3}$$

where

$$\tilde{w}_i(\lambda; y_{i+1}, y_i, x_i) = \frac{y_i^{-1}}{\Gamma(c_i - \lambda)} \left(\frac{x_i - by_{i+1}}{y_i} \right)^{c_i - \lambda - 1} \exp\left(\frac{by_{i+1} - x_i}{y_i} \right) \tag{6.4}$$

and integration in x_i is taken over the straight ray starting from $x_i = by_{i+1}$ and extending to infinity in the $y_i/|y_i|$ direction.

Note that the kernels $\mathcal{Q}_\lambda(\vec{y} \mid \vec{x})$ and $\tilde{\mathcal{Q}}_\lambda(\vec{y} \mid \vec{x})$ satisfy the semiclassical condition (4.19) which, taking into account our quantization convention $-\mathrm{i}\hbar = 1$, takes the following form (up to insignificant λ-dependent factors):

$$\mathcal{Q}_\lambda(\vec{y} \mid \vec{x}) \simeq \exp\bigl(-F_\lambda(\vec{y} \mid \vec{x})\bigr) \tag{6.5}$$

with the generating function of the Bäcklund transformation given by (2.14). Actually, the semiclassical approximation is almost exact, up to a minor quantum correction $c_i - \lambda \mapsto c_i - \lambda - 1$. This fact supports our thesis on the intermediate position of the DST model, with regard to complexity, between the Toda lattice and the generic XXX spin chain. For comparison, in the case of the Toda lattice the semiclassical formula for $\mathcal{Q}_\lambda(\vec{y} \mid \vec{x})$ is plainly exact [12], whereas for the XXX spin chain there is little hope of such a simple result.

For the purposes of the present section we need the expressions for Q_λ similar to formulae (5.10) and (5.11) for $\mathbb{R}_\lambda$. The corresponding formulae are produced, respectively, from (6.1) and (6.3) by the change of variables $x_i = y_i\xi_i + by_{i+1}$.

The analogue of (5.10) is the formula

$$Q_\lambda : \psi(\vec{x}) \mapsto \int_\gamma \mathrm{d}\xi_1 \ldots \int_\gamma \mathrm{d}\xi_n\, \mathcal{W}_\lambda(\vec{\xi})\psi(\ldots, y_i\xi_i + by_{i+1}, \ldots) \tag{6.6}$$

where

$$\mathcal{W}_\lambda(\vec{\xi}) = \prod_{i=1}^{n} \frac{\mathrm{i}}{2\pi}\Gamma(\lambda + 1 - c_i)\mathrm{e}^{-\xi_i}(-\xi_i)^{c_i-\lambda-1} \tag{6.7}$$

valid for any complex λ, except the poles $\lambda = c_i - k$, $(i = 1, \ldots, n;\ k = 1, 2, \ldots)$ of $\Gamma(\lambda + 1 - c_i)$. The branch of each of many-valued functions $(-\xi_i)^{c_i-\lambda-1}$ in (6.6) is fixed by making a cut along $(0, \infty)$ and assuming that $-\pi \leqslant \arg(-\xi_i) \leqslant \pi$.

The analogue of (5.11) is the formula

$$Q_\lambda : \psi(\vec{x}) \mapsto \int_0^\infty \mathrm{d}\xi_1 \ldots \int_0^\infty \mathrm{d}\xi_n\, \tilde{\mathcal{W}}_\lambda(\vec{\xi})\psi(\ldots, y_i\xi_i + by_{i+1}, \ldots) \tag{6.8}$$

with the kernel $\tilde{\mathcal{W}}_\lambda$

$$\tilde{\mathcal{W}}_\lambda(\vec{\xi}) = \prod_{i=1}^{n} \frac{\mathrm{e}^{-\xi_i}\xi_i^{c_i-\lambda-1}}{\Gamma(c_i - \lambda)} \tag{6.9}$$

valid for $\operatorname{Re}\lambda < \min \operatorname{Re} c_i$. Together, formulae (6.6) and (6.8) define Q_λ as a holomorphic function of $\lambda \in \mathbb{C}$.

In the rest of this section we will show, that Q_λ maps polynomials in x into polynomials in y and λ, and derive explicit formulae for its action on the monomial basis in $\mathbb{C}[\vec{x}]$.

Before considering the general case we will give a brief account of the simplest $n = 1$ case. In this case we have only one variable $x \equiv x_1$, the Lax matrix simplifies to $L(u) = \ell(u)$, so, without loss of generality, one can put $c_1 = 0$. The trace of $L(u)$ gives rise to only one integral of motion (number of particles N)

$$t(u) \equiv \operatorname{tr} L(u) = u - N + b \qquad N = x\partial. \tag{6.10}$$

We assume that N acts in the space $\mathbb{C}[x]$ of polynomials of x spanned by the eigenbasis $\{x^m\}_{m=0}^\infty$ of N

$$N : x^m \mapsto m x^m \qquad m = 0, 1, 2, \ldots. \tag{6.11}$$

For $n = 1$ and $c_1 = 0$ formula (6.6) defining the Q operator turns into

$$Q_\lambda : \psi(x) \mapsto \frac{\mathrm{i}}{2\pi}\Gamma(\lambda + 1)\int_\gamma \mathrm{d}\xi\, \mathrm{e}^{-\xi}(-\xi)^{-\lambda-1}\psi(y(\xi + b)) \qquad \lambda \neq -1, -2, \ldots \tag{6.12}$$

and (6.8), respectively, into

$$Q_\lambda : \psi(x) \mapsto \frac{1}{\Gamma(-\lambda)}\int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-\xi}\xi^{-\lambda-1}\psi(y(\xi + b)) \qquad \operatorname{Re}\lambda < 0. \tag{6.13}$$

Similarly, from (6.1) and (6.3) one gets, respectively,

$$Q_\lambda : \psi(x) \mapsto \frac{\mathrm{i}\mathrm{e}^b}{2\pi}\Gamma(\lambda + 1)\int_{\gamma'} \mathrm{d}x\, y^{-1}\left(b - \frac{x}{y}\right)^{-\lambda-1}\mathrm{e}^{-x/y}\psi(x) \qquad \gamma' = y(\gamma + b) \tag{6.14}$$

and

$$Q_\lambda : \psi(x) \mapsto \frac{\mathrm{e}^b}{\Gamma(-\lambda)} \int_{by}^{\infty} \mathrm{d}x\, y^{-1} \left(\frac{x}{y} - b\right)^{-\lambda-1} \mathrm{e}^{-x/y} \psi(x) \qquad y > 0. \tag{6.15}$$

To calculate explicitly the action of Q_λ on the basis x^m one puts $\psi(x) = x^m$ in (6.12), then expands the binomial $(\xi + b)^m$ and applies, termwise, Hankel's integral formula (5.12). This calculation is very similar to the calculation of $\mathbb{R}_\lambda s^k x^j$ given by formula (5.13). The result is that the monomials $\{x^m\}_{m=0}^{\infty}$ are the eigenvectors of Q_λ:

$$Q_\lambda : x^m \mapsto q_m(\lambda) y^m \tag{6.16}$$

the corresponding eigenvalues $q_m(\lambda)$ being polynomials in λ of degree m, expressed in terms of the Charlier polynomials $C_m(\lambda; b)$ as

$$q_m(\lambda) = \sum_{j=0}^{m} \binom{m}{j} (-\lambda)_j b^{m-j} = b^m C_m(\lambda; b) \tag{6.17}$$

(cf (5.13)).

As an immediate consequence, we have the commutativity $[Q_\lambda, N] = 0$, as well as (4.17*a*) and (4.17*b*). Another corollary is that Q_λ maps $\mathbb{C}[x]$ into $\mathbb{C}[y, \lambda]$. Note that formula (6.17) implies the normalization $Q_\lambda : 1 \mapsto 1$.

One can use the integral operator Q_λ to derive a few well known formulae for the orthogonal Charlier polynomials. For instance, putting $\psi(x) = x^m$ and $y = 1$ in (6.12) or (6.13) one obtains integral representations for Charlier polynomials:

$$C_m(\lambda; b) = \frac{\mathrm{i}}{2\pi} \Gamma(\lambda + 1) \int_\gamma \mathrm{d}\xi\, \mathrm{e}^{\xi} (-\xi)^{-\lambda-1} \left(1 + \frac{\xi}{b}\right)^m \tag{6.18}$$

and, respectively,

$$C_m(\lambda; b) = \frac{1}{\Gamma(-\lambda)} \int_0^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-\xi} \xi^{-\lambda-1} \left(1 + \frac{\xi}{b}\right)^m \tag{6.19}$$

(see [22]).

Equating $\psi(x)$ in (6.12) or (6.13) with the generating function $\mathrm{e}^{tx} = \sum_{m=0}^{\infty} x^m t^m / m!$ of the monomials x^m and taking the integral one gets the generating function of Charlier polynomials

$$\mathrm{e}^t \left(1 - \frac{t}{b}\right)^{\lambda} = \sum_{m=0}^{\infty} \frac{t^m}{m!} C_m(\lambda; b). \tag{6.20}$$

The recurrence relation for the Charlier polynomials [22, 23] is equivalent to the finite-difference equation for the polynomials $q_i(\lambda)$:

$$(\lambda - i + b) q_i(\lambda) = b q_i(\lambda + 1) + \lambda q_i(\lambda - 1) \tag{6.21}$$

which coincides with Baxter's equation (4.14) for $n = 1$ and proves, for $n = 1$, the operator relation (4.17*c*).

From the explicit expression (6.17) for the polynomials $q_m(\lambda)$ we conclude that they are normalized by the condition $q_m(0) = b^m$, or, alternatively, $q_m(\lambda) = (-\lambda)^m + \mathrm{O}(\lambda^{m-1})$, as $\lambda \to \infty$. In terms of the operator Q_λ, it is equivalent to

$$Q_0 = b^N \tag{6.22}$$

(see (6.10) for the definition of N) and, respectively, to

$$Q_\lambda = (-\lambda)^N + \mathrm{O}(\lambda^{N-1}). \tag{6.23}$$

The generalization of the above results to the multivariable case is quite straightforward. To calculate explicitly the action of Q_λ on the monomial basis $x_1^{m_1} \ldots x_n^{m_n}$ in $\mathbb{C}[\vec{x}]$ one substitutes

$\psi(\vec{x}) = x_1^{m_1} \ldots x_n^{m_n}$ into (6.6), then expands the binomials $(y_i\xi_i + by_{i+1})^{m_i}$ and uses, termwise, Hankel's integral formula (5.12). Recalling definition (6.17) of Charlier polynomials, one obtains the following expression:

$$Q_\lambda : x_1^{m_1} \ldots x_n^{m_n} \mapsto \prod_{i=1}^{n} b^{m_i} y_{i+1}^{m_i} C_{m_i}(\lambda - c_i; by_{i+1}/y_i) \tag{6.24}$$

from which it follows immediately that the normalization condition $Q_\lambda : 1 \mapsto 1$ holds and that Q_λ maps $\mathbb{C}[\vec{x}]$ into $\mathbb{C}[\vec{y}, \lambda]$. The polynomiality of matrix elements of Q_λ combined with the commutativity $[Q_{\lambda_1}, Q_{\lambda_2}]$ (4.17*b*) proves the polynomiality (4.17*d*) of the eigenvalues of Q_λ.

Formula (6.24) also allows one to determine the normalization (4.16) of the eigenvalues of Q_λ. Taking the limit $\lambda \to \infty$ in (6.24) and using the asymptotics $C_m(\lambda; b) = (-\lambda/b)^m + \mathrm{O}(\lambda^{m-1})$ we conclude that, as in the $n = 1$ case, Q_λ has the asymptotics (6.23) with the operator N given by (1.2). In contrast, equality (6.22), generally speaking, cannot be generalized to $n > 1$, with the exception of the homogeneous chain case $c_i \equiv 0$, $i = 1, \ldots, n$, when it is replaced by

$$Q_0 = b^N U \tag{6.25}$$

where U is the translation operator $U : x_i \to y_{i+1}$.

As a final remark of this section, we point out yet another way of expressing the action of Q_λ. Substituting $\psi(\vec{x})$ in (6.6) with the polynomials $\omega_{\vec{y}}^{\vec{m}} \in \mathbb{C}[\vec{x}]$

$$\omega_{\vec{y}}^{\vec{m}}(\vec{x}) = \prod_{i=1}^{n} (x_i - by_{i+1})^{m_i}$$

parametrized by the multi-index $\vec{m} = (m_1, \ldots, m_n)$ and a vector $\vec{y} = (y_1, \ldots, y_n)$ we obtain, after performing the integrations, an elegant formula for the action of Q_λ on $\omega_{\vec{y}}^{\vec{m}}$:

$$Q_\lambda : \prod_{i=1}^{n} (x_i - by_{i+1})^{m_i} \mapsto \prod_{i=1}^{n} (c_i - \lambda)_{m_i} y_i^{m_i}. \tag{6.26}$$

Formula (6.26) seems to provide the most compact way to encode the action of Q_λ on polynomials (compare with formula (5.14) for the action of $\mathbb{R}_\lambda$). Some caution is necessary, however, when using it, since the parameters $\vec{y}$ in $\omega_{\vec{y}}^{\vec{m}}$ coincide with the variables in the target space $\mathbb{C}[\vec{y}]$ of Q_λ. One way of interpreting (6.26) is to consider its left-hand side as a short-hand notation for $[Q_\lambda \omega_{\vec{z}}^{\vec{m}}]_{\vec{z}=\vec{y}}$. Another possibility is to extend the operator Q_λ onto the polynomial ring $\mathbb{C}[\vec{x}, \vec{y}]$ assuming that it acts on $\vec{y}$ trivially: $Q_\lambda(\psi(x)\varphi(y)) = \varphi(y)Q_\lambda(\psi(x))$. Formulae similar to (6.26) also arise in the separation of variables for Macdonald polynomials [24].

It is a challenging problem to take formulae (5.14) and (6.26) as definitions of $\mathbb{R}_\lambda$ and Q_λ, respectively, and to build the theory of Q_λ in an entirely algebraic way.

7. Baxter's equation

In the previous sections we have proved all the properties of Q_λ from the list given in section 4 except Baxter's difference equation (4.17*c*). In this section we give a proof of identity (4.17*c*) based on the ideas of [12].

For our purposes, the best suited realization of Q_λ is that given by formulae (4.18) and (6.1). Recalling that $t(u) = \operatorname{tr} L(u)$ and that $L(u)$ is a 2×2 matrix whose entries are differential operators in x_i, we can transform the left-hand side of (4.17*c*) as follows:

$$[Q_\lambda t(\lambda)\psi](\vec{y}) = \operatorname{tr}[Q_\lambda L(\lambda)\psi](\vec{y}) = \operatorname{tr} \int \mathrm{d}x^n \, \mathcal{Q}_\lambda(\vec{y} \mid \vec{x}) L(\lambda)\psi(\vec{x}).$$

Performing integration by parts we obtain

$$[Q_\lambda t(\lambda)\psi](\vec{y}) = \operatorname{tr}\int \mathrm{d}x^n\, [L^*(\lambda)\mathcal{Q}_\lambda(\vec{y} \mid \vec{x})]\psi(\vec{x}) \tag{7.1}$$

where $L^*(\lambda)$ is the matrix composed of adjoint differential operators $(L_{jk})^* = L^*_{jk}$. For example, $\partial^* = -\partial$, $(x\partial)^* = -\partial x = -x\partial - 1$, and so on.

Using the factorization (2.2) of $L(\lambda)$ into the product of elementary Lax matrices $\ell_i(\lambda)$ and the factorization (6.1) of the kernel $\mathcal{Q}_\lambda(\vec{y} \mid \vec{x})$ into the factors w_i (6.2), we can represent the kernel of the integral operator $Q_\lambda t(\lambda)$ as

$$[Q_\lambda t(\lambda)](\vec{y} \mid \vec{x}) = \operatorname{tr} \ell^*_n(\lambda)\ldots\ell^*_1(\lambda)\prod_{i=1}^n w_i = \operatorname{tr}(\ell^*_n(\lambda)w_n)\ldots(\ell^*_1(\lambda)w_1) \tag{7.2}$$

where

$$\ell^*_i(\lambda) = \begin{pmatrix} \lambda - c_i + 1 + x_i\partial_{x_i} & bx_i \\ \partial_{x_i} & b \end{pmatrix}. \tag{7.3}$$

The possibility of the factorization (7.2) of $[Q_\lambda L(\lambda)](\vec{y} \mid \vec{x})$ depends crucially on the fact that the factors w_i (6.2) each depend only on one variable x_i. That is why we take the left-hand side of (4.17*c*) to be $Q_\lambda t(\lambda)$ rather than $t(\lambda)Q_\lambda$.

The rest of the proof parallels that of the spectrality property for the classical case given in section 2. Introducing matrices $\tilde{\ell}_i(\lambda)$ by the equality $\ell^*_i(\lambda)w_i = w_i\tilde{\ell}_i(\lambda)$ and noting that

$$\partial_{x_i}\ln w_i(y_{i+1}, y_i, x_i) = \frac{c_i - \lambda - 1}{x_i - by_{i+1}} - \frac{1}{y_i} \tag{7.4}$$

we obtain

$$\tilde{\ell}_i(\lambda) = \begin{pmatrix} \lambda - c_i + 1 + x_i\partial_{x_i}\ln w_i & bx_i \\ \partial_{x_i}\ln w_i & b \end{pmatrix} = \begin{pmatrix} \frac{b(c_i-\lambda-1)y_{i+1}}{x_i-by_{i+1}} - \frac{x_i}{y_i} & bx_i \\ \frac{c_i-\lambda-1}{x_i-by_{i+1}} - \frac{1}{y_i} & b \end{pmatrix} \tag{7.5}$$

and

$$\big[Q_\lambda t(\lambda)\big](\vec{y} \mid \vec{x}) = \mathcal{Q}_\lambda(\vec{y} \mid \vec{x}) \operatorname{tr}\tilde{\ell}_n(\lambda)\ldots\tilde{\ell}_1(\lambda) \equiv \mathcal{Q}_\lambda(\vec{y} \mid \vec{x})\operatorname{tr}\tilde{L}(\lambda). \tag{7.6}$$

We note then that the gauge transformation $\tilde{\ell}_i(\lambda) \mapsto N^{-1}_{i+1}\tilde{\ell}_i(\lambda)N_i$ with the gauge matrix

$$N_i = \begin{pmatrix} 1 & by_i \\ 0 & 1 \end{pmatrix} \tag{7.7}$$

leaves $\operatorname{tr}\tilde{L}(\lambda)$ invariant while making $\tilde{\ell}_i(\lambda)$ and, consequently, $\tilde{L}(\lambda)$ triangular:

$$\begin{aligned} N^{-1}_{i+1}\tilde{\ell}_i(\lambda)N_i &= \begin{pmatrix} -\frac{x_i-by_{i+1}}{y_i} & 0 \\ \frac{c_i-\lambda-1}{x_i-by_{i+1}} - \frac{1}{y_i} & \frac{b(c_i-\lambda-1)y_i}{x_i-by_{i+1}} \end{pmatrix} \\ &= \begin{pmatrix} (\lambda - c_i)\frac{w_i(\lambda-1)}{w_i(\lambda)} & 0 \\ \frac{c_i-\lambda-1}{x_i-by_{i+1}} - \frac{1}{y_i} & \frac{bw_i(\lambda+1)}{w_i(\lambda)} \end{pmatrix} \end{aligned} \tag{7.8}$$

where we used the identities

$$\frac{w_i(\lambda+1)}{w_i(\lambda)} = \frac{(c_i - \lambda - 1)y_i}{x_i - by_{i+1}} \qquad \frac{w_i(\lambda-1)}{w_i(\lambda)} = \frac{x_i - by_{i+1}}{(c_i - \lambda)y_i}. \tag{7.9}$$

As a result, we get the equality

$$\operatorname{tr}\tilde{L}(\lambda) = b^n\prod_{i=1}^n \frac{w_i(\lambda+1)}{w_i(\lambda)} + \prod_{i=1}^n(\lambda - c_i)\frac{w_i(\lambda-1)}{w_i(\lambda)} \tag{7.10}$$

which, obviously, proves (4.17*c*).

8. Discussion

For the example of the quantum integrable DST model we have shown that the construction of the Q-operator as an integral operator, in the style of [12], and as the trace of a monodromy matrix with a special representation of the quantum group corresponding to the auxiliary space, in the style of [14], can be combined naturally within an unified approach. The same approach can be applied to other integrable models which are more general than the DST model, such as the generic XXX magnetic chain. This work is in progress and the results will be reported in a separate paper. For a particular case of the homogeneous XXX chain a Q-operator was recently constructed in [25].

Another interesting problem is to build the theory of the Q-operator in a purely algebraic manner starting from formulae (5.14) and (6.26).

In [14] it is argued that for the models governed by the A_1-type R-matrices there exist two different Q-operators corresponding to two different q-oscilator representations of $\mathcal{U}_q[\widehat{sl_2}]$. Their eigenvalues correspond, respectively, to two linearly independent solutions of Baxter's difference equations analogous to (4.14). In the case of the DST model the second Q-operator can be obtained if we choose, in formula (5.4), another representation $\mathcal{M}(u-\lambda)$ of the algebra (5.5), namely $\tilde{\mathcal{M}}(u-\lambda) \sim -\mathcal{M}^{-1}(\lambda-u)$

$$\tilde{\mathcal{M}}(u; s, \partial_s) = \begin{pmatrix} -1 & s \\ -\partial_s & u + s\partial_s \end{pmatrix}. \tag{8.1}$$

The corresponding Q-operator has, however, more complex nature than the one studied in this paper. Its eigenvalues, for example, are not polynomial in λ. The problem is currently under study.

We can point out the following application of our results to the theory of special functions of many variables. Notice that the eigenfunctions of the quantum DST Hamiltonians are multivariable polynomials. The family of integral equations obtained for those polynomials provided by the Q_λ-operator supplements their representation as Bethe vectors and can be used in efficient numerical calculations of these special functions, for instance, solving integral equations by iterations. Simple considerations of the $n = 2$ case show that we deal with multivariable analogues of the Heun polynomials. For special functions of such complexity the integral equations found might be the only explicit representations to exist because there is no hope to get, for instance, an integral representation. So, the integral equations found for the special functions which were initially defined as eigenfunctions of the commuting differential operators can be used first, as already remarked, for generating advanced numerical methods of their calculation, and, secondly, for finding various asymptotics. These applications are being worked on.

Acknowledgments

The authors wish to acknowledge the support of EPSRC and INTAS.

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